HOMEWORK 8 SOLUTIONS AND 4.3, 4.4 EXs

Section 4.1 : #15a

Let $f: \mathbb{Z}_4 \to \mathbb{Z}_{12}$

$$[x]_4 \mapsto [3x]_{12}.$$

Show f well-defined.

Proof: We must show equivalent inputs to f give equivalent outputs.

Suppose $x \equiv y \mod 4$ and we must show $3x \equiv 3y \mod 12$.

Since $x \equiv y \mod 4$ we know $4 \mid y - x$

so $\exists\, k \in \mathbb{Z}$ with $4k = y - x$

$$\Rightarrow 3(4k) = 3(y - x)$$

$$\Rightarrow 12k = 3y - 3x$$

$$\Rightarrow 12 \mid 3y - 3x$$

$$\Rightarrow 3y \equiv 3x \mod 12. \quad \blacksquare$$

Section 4.3 #1c

Let $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$
$x \mapsto (x, x)$.

Is f onto? Prove your answer.

f is not onto.

Proof: Notice $(1,2) \in \mathbb{N} \times \mathbb{N}$ is in the codomain,
but $f(x) = (1,2) \Rightarrow (x,x) = (1,2)$
$\Rightarrow x = 1$ and $x = 2$
which is impossible, so $(1,2) \notin Rng(f)$.
Thus f is not onto. ∎

Section 4.3 #2c

Let f be as above. Is f one-to-one? Prove answer.

f is one-to-one

Proof: Let $x, y \in \mathbb{N}$ be such that $f(x) = f(y)$.
Then $(x,x) = (y,y)$ so $x = y$. ∎

Section 4.3 #5

If $f: A \xrightarrow{\text{onto}} B$ and $g: B \xrightarrow{\text{onto}} C$

then $g \circ f: A \xrightarrow{\text{onto}} C$.

Proof: To show $g \circ f$ is onto let $c \in C$ be arbitrary. Since $g: B \to C$ is onto $\exists$ $b \in B$ with $g(b) = c$.

Since $f: A \to B$ is onto $\exists$ $a \in A$ with $f(a) = b$.

Then $g \circ f(a) = g(f(a)) = g(b) = c$. ☒

Section 4.3 #6

If $f: A \to B$, $g: B \to C$, and $g \circ f: A \xrightarrow{1-1} B$

then f is one-to-one.

Proof: To show f one-to-one let $x, y \in A$ with $f(x) = f(y)$.

Applying g to both sides gives

$$g(f(x)) = g(f(y)) \Rightarrow g \circ f(x) = g \circ f(y).$$

Since $g \circ f$ is one-to-one by assumption this implies $x = y$, as desired. ☒

Section 4.4: #19

Let $f : \mathbb{Z}_8 \to \mathbb{Z}_8$
$\qquad \bar{x} \longmapsto \overline{5x-1}$.

Show f is a bijection.

Proof: To show f is onto let $\bar{y} \in \mathbb{Z}_8$.

Define $\bar{z} = \overline{5(y+1)} \in \mathbb{Z}_8$. Then

$$f(\bar{z}) = \overline{5z-1} = \overline{5 \cdot 5(y+1) - 1}$$
$$= \overline{25}\left(\overline{y+1}\right) - \bar{1}$$
$$= \overline{y+1-1} = \bar{y} \quad \left(\text{using the fact that } \overline{25} = \bar{1}.\right)$$

Thus, f is onto.

To show f one-to-one, let $\bar{x}, \bar{y} \in \mathbb{Z}_8$ with $f(\bar{x}) = f(\bar{y})$

$$\Rightarrow \overline{5x-1} = \overline{5y-1}$$
$$\Rightarrow \overline{5x} = \overline{5y}$$
$$\Rightarrow \bar{5}\left(\overline{5x}\right) = \bar{5}\left(\overline{5y}\right)$$
$$\Rightarrow \overline{25x} = \overline{25y}$$
$$\Rightarrow \bar{x} = \bar{y}$$

again using the fact that $\overline{25} = \bar{1}$. ∎

Section 4.4 #4

If $f: A \to B$ and f^{-1} is a function, then f is injective (one-to-one)

Proof: Let $x, y \in A$ with $f(x) = f(y)$.

and let $z = f(x) \in B$.

Then $f(x) = z$, $f(y) = z \Rightarrow (x, z), (y, z) \in f$

$$\Rightarrow (z, x), (z, y) \in f^{-1}.$$

Since f^{-1} is a function by assumption it is single valued, so $x = y$. ∎